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Lattice Formulation of 2D SQCD with exact supersymmetry

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1 Introduction

◇ Lattice Formulation : Standard method for nonperturbative study of QFT [Wilson]

Fewer relevant operators to be fine-tuned in the continuum limit

$\longrightarrow$ Practically easier to construct the continuum theory

$\Rightarrow$ Desirable to possess more symmetries at the level of lattice theory

◇ Extension to SUSY Theories

A part of supercharges can be preserved on the lattice

2D Wess-Zumino model [Sakai-Sakamoto, Kikukawa-Nakayama, Catterall]

pure SYM models [Kaplan et al] $\longleftarrow$ orbifolding, [F.S., Catterall] $\longleftarrow$ TFT approach

SYM + matter fields [Endre-Kaplan, Matsuura] $\longleftarrow$ orbifolding, [F.S.] This Talk $\longleftarrow$ TFT approach

> Here, we construct a lattice theory for
> 2D $\mathcal{N} = (2, 2)$ SQCD (SYM + (anti-)fundamental matters)
> with $G = U(N)$ (or $SU(N)$)
> compact gauge fields
> general matter superpotential,
> keeping one of the supercharge Q.

2 SYM part of the Lattice Action

4D $\mathcal{N}=1$ SYM $\Rightarrow$ (dim. red.) $\Rightarrow$ 2D $\mathcal{N}=(2,2)$ SYM

$A_\mu \quad (\mu=1,2)$ $\qquad$ $A_\mu \Rightarrow U_\mu(x)$ (link variables on the lattice)

A_3, A_4 $\qquad$ $\phi(x), \bar{\phi}(x)$ (site variables)

Fermions : 4-component Majorana spinor

$\Psi(x) = \left(\psi_1(x), \psi_2(x), \chi(x), \frac{1}{2}\eta(x)\right)^T$ (site variables)

<u>Q-SUSY on the lattice</u>

For admissible gauge fields ($||1-U_{12}(x)|| < \epsilon$)

$$\begin{aligned}
&QU_\mu(x) = i\psi_\mu(x)U_\mu(x)\\
&Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) + iD_\mu\phi(x)\\
&Q\phi(x) = 0\\
&Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)]\\
&Q\chi(x) = iD(x) + \frac{i}{2}\widehat{\Phi}(x), \quad QD(x) = -\frac{1}{2}Q\widehat{\Phi}(x) - i[\phi(x), \chi(x)],
\end{aligned} \tag{2.1}$$

where $D_\mu\phi(x) = U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^{-1} - \phi(x)$ (covariant difference) ,

$\widehat{\Phi}(x) = \frac{-i(U_{12}(x)-U_{21}(x))}{1-\frac{1}{\epsilon^2}||1-U_{12}(x)||^2} \sim 2F_{12}$

Note

The admissibility and the denominator smoothly remove the degenerated vacua $U_{12}(x)^2 = 1, U_{12}(x) \neq 1$. [Lüscher],

(Take the traceless part of the numerator for $G = SU(N)$ case)

$\Rightarrow Q^2 =$ (infinitesimal gauge tr. with the parameter $\phi(x)$)

Lattice Action: Q-exact form $\Rightarrow$ Exact Q-SUSY

For admissible gauge fields ($||1 - U_{12}(x)|| < \epsilon$ for $\forall x$),

$$\begin{aligned} S_{\rm SYM}^{\rm (lat)} &= Q\frac{1}{g^2}\sum_x \mathrm{tr}\left[\chi(x)\left\{-\frac{i}{2}\widehat{\Phi}(x) + iD(x)\right\} + \frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\sum_\mu \psi_\mu(x)D_\mu\bar{\phi}(x)\right] \\ &= \frac{1}{g^2}\sum_x \mathrm{tr}\left[\frac{1}{4}\widehat{\Phi}(x)^2 + \sum_\mu D_\mu\phi(x)D_\mu\bar{\phi}(x) + i\chi(x)Q\widehat{\Phi}(x) + i\sum_\mu \psi_\mu(x)D_\mu\eta(x)\right. \\ &\quad + \frac{1}{4}[\phi(x),\bar{\phi}(x)]^2 - \chi(x)[\phi(x),\chi(x)] - \frac{1}{4}\eta(x)[\phi(x),\eta(x)] \\ &\quad \left. - \sum_\mu \psi_\mu(x)\psi_\mu(x)\left(\bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^{-1}\right) - D(x)^2\right], \end{aligned} \tag{2.2}$$

For other cases, $S_{\rm SYM}^{\rm (lat)} = +\infty$. (i.e. The Boltzmann weight is zero.)

Note

Gauge field configurations are smoothly cut into the admissible ones.

c.f. $f(t) = \begin{cases} e^{-c/t} & t \geq 0 \\ 0 & t < 0 \end{cases}$ with $c > 0$ is smooth and infinitely differentiable w.r.t. $t \in \mathrm{R}$

$\Rightarrow$ Q-SUSY is preserved.

The Q-SUSY forbids the mass term $\phi\bar{\phi}$ appearing in the radiative correction.
$\Rightarrow$ The continuum theory can be constructed without any fine-tuning.
(Checked in the lattice perturbation. Computer simulation will give the nonperturbative check.)

3 Matter part of the Lattice Action

Φ_{+I} : Dim. Red. of 4D $\mathcal{N}=1$ chiral superfield (fundamental repre.)
(Flavors: $I=1,\cdots,n_+$)
Φ_{-I} : Dim. Red. of 4D $\mathcal{N}=1$ chiral superfield (anti-fundamental repre.)
(Flavors: $I=1,\cdots,n_-$)
The continuum theory is

$$\mathcal{L}_{\rm mat} = \left[\sum_{I=1}^{n_+}\Phi^\dagger_{+I}e^{V-\widetilde{V}_{+I}}\Phi_{+I}+\sum_{I=1}^{n_-}\Phi_{-I}e^{-V+\widetilde{V}_{-I}}\Phi^\dagger_{-I}\right]\Bigg|_{\theta\theta\bar\theta\bar\theta} \tag{3.1}$$

$$\mathcal{L}_{\rm pot} = W(\Phi_+,\Phi_-)|_{\theta\theta}+\overline{W}(\Phi^\dagger_+,\Phi^\dagger_-)\Big|_{\bar\theta\bar\theta} \tag{3.2}$$

where $\widetilde{V}_{\pm I}\equiv 2\theta_R\bar\theta_L\widetilde{m}_{\pm I}+2\theta_L\bar\theta_R\widetilde{m}^*_{\pm I}$: twisted masses.

Note

Two kinds of fermion mass can be introduced.

- Complex mass terms $(\subset W,\overline{W})$:
  $m_I\left(\psi_{-IL}\psi_{+IR}-\psi_{-IR}\psi_{+IL}\right)+m^*_I\left(\bar\psi_{+IR}\bar\psi_{-IL}-\bar\psi_{+IL}\bar\psi_{-IR}\right)$
- Twisted mass terms $(\not\subset W,\overline{W})$:
  $\widetilde{m}_{+I}\bar\psi_{+IL}\psi_{+IR}+\widetilde{m}^*_{+I}\bar\psi_{+IR}\psi_{+IL}+\widetilde{m}_{-I}\psi_{-IR}\bar\psi_{-IL}+\widetilde{m}^*_{-I}\psi_{-IL}\bar\psi_{-IR}$

◇ Flavor symmetry of $\mathcal{L}_{\rm mat}$: $U(n_+)\times U(n_-)$ $(\widetilde{m}_{\pm I}=\widetilde{m}^*_{\pm I}=0$ case)

$\Rightarrow U(1)^{n_+} \times U(1)^{n_-}$ (generic $\widetilde{m}_{\pm I}$, $\widetilde{m}^*_{\pm I}$ case)

◇ Let us consider the latticization, introducing the forward (backward) covariant differences $D_\mu (D^*_\mu)$:

$$
\begin{aligned}
D_\mu \Phi_{+I}(x) &\equiv U_\mu(x)\Phi_{+I}(x+\hat{\mu}) - \Phi_{+I}(x) \\
D^*_\mu \Phi_{+I}(x) &\equiv \Phi_{+I}(x) - U_\mu(x-\hat{\mu})^{-1}\Phi_{+I}(x-\hat{\mu}) \\
D_\mu \Phi_{-I}(x) &\equiv \Phi_{-I}(x+\hat{\mu})U_\mu(x)^{-1} - \Phi_{-I}(x) \\
D^*_\mu \Phi_{-I}(x) &\equiv \Phi_{-I}(x) - \Phi_{-I}(x-\hat{\mu})U_\mu(x-\hat{\mu}) \\
&\vdots
\end{aligned}
\tag{3.3}
$$

<u>Q-SUSY on the lattice</u> [Consider the case $n_+ = n_- \equiv n$]

$$
\begin{aligned}
&Q\phi_{+I}(x) = -\psi_{+IL}(x), \quad Q\psi_{+IL}(x) = -(\phi(x) - \widetilde{m}_{+I})\phi_{+I}(x) \\
&Q\phi^\dagger_{+I}(x) = -\bar{\psi}_{+IR}(x), \quad Q\bar{\psi}_{+IR}(x) = \phi_{+I}(x)^\dagger(\phi(x) - \widetilde{m}_{+I}) \\
&Q\psi_{+IR}(x) = \left\{\frac{1}{2}(D_1 + D^*_1) + i\frac{1}{2}(D_2 + D^*_2)\right\}\phi_{+I}(x) + F_{+I}(x) \\
&\qquad + \sum_\mu \frac{r}{2}(D_\mu - D^*_\mu)\phi_{-I}(x)^\dagger \quad \leftarrow \text{Wilson term } (r > 0) \\
&Q\bar{\psi}_{+IL}(x) = \left\{\frac{1}{2}(D_1 + D^*_1) - i\frac{1}{2}(D_2 + D^*_2)\right\}\phi_{+I}(x)^\dagger + F_{+I}(x)^\dagger
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\sum_\mu \frac{r}{2}(D_\mu - D_\mu^*)\phi_{-I}(x) \\
QF_{+I}(x) &= (\phi(x) - \widetilde{m}_{+I})\psi_{+IR}(x) + \left\{\frac{1}{2}(D_1 + D_1^*) + i\frac{1}{2}(D_2 + D_2^*)\right\}\psi_{+IL}(x) \\
&\quad + \sum_\mu \frac{r}{2}(D_\mu - D_\mu^*)\bar{\psi}_{-IR}(x) \\
&\quad - i\frac{1}{2}\left(\psi_1(x)U_1(x)\phi_{+I}(x+\hat{1}) + U_1(x-\hat{1})^{-1}\psi_1(x-\hat{1})\phi_{+I}(x-\hat{1})\right) \\
&\quad + \frac{1}{2}\left(\psi_2(x)U_2(x)\phi_{+I}(x+\hat{2}) + U_2(x-\hat{2})^{-1}\psi_2(x-\hat{2})\phi_{+I}(x-\hat{2})\right) \\
&\quad - i\sum_\mu \frac{r}{2}\left(\psi_\mu(x)U_\mu(x)\phi_{-I}(x+\hat{\mu})^\dagger - U_\mu(x-\hat{\mu})^{-1}\psi_\mu(x-\hat{\mu})\phi_{-I}(x-\hat{\mu})^\dagger\right), \\
QF_{+I}(x)^\dagger &= -\bar{\psi}_{+IL}(x)(\phi(x) - \widetilde{m}_{+I}) + \left\{\frac{1}{2}(D_1 + D_1^*) - i\frac{1}{2}(D_2 + D_2^*)\right\}\bar{\psi}_{+IR}(x) \\
&\quad + \sum_\mu \frac{r}{2}(D_\mu - D_\mu^*)\psi_{-IL}(x) \\
&\quad + i\frac{1}{2}\left(\phi_{+I}(x+\hat{1})^\dagger U_1(x)^{-1}\psi_1(x) + \phi_{+I}(x-\hat{1})^\dagger\psi_1(x-\hat{1})U_1(x-\hat{1})\right) \\
&\quad + \frac{1}{2}\left(\phi_{+I}(x+\hat{2})^\dagger U_2(x)^{-1}\psi_2(x) + \phi_{+I}(x-\hat{2})^\dagger\psi_2(x-\hat{2})U_2(x-\hat{2})\right) \\
&\quad + i\sum_\mu \frac{r}{2}\left(\phi_{-I}(x+\hat{\mu})U_\mu(x)^{-1}\psi_\mu(x) - \phi_{-I}(x-\hat{\mu})\psi_\mu(x-\hat{\mu})U_\mu(x-\hat{\mu})\right), \\
&\vdots
\end{aligned}
\tag{3.4}
$$

$$\Rightarrow Q^2 = (\text{infinitesimal gauge tr. with the parameter } \phi(x)) + (\text{infinitesimal flavor rotation with the parameter } \widetilde{m}_{\pm I}) \quad (3.5)$$

$$\delta\Phi_{+I} = -\widetilde{m}_{+I}\Phi_{+I}, \quad \delta\Phi^{\dagger}_{+I} = \widetilde{m}_{+I}\Phi^{\dagger}_{+I}$$
$$\delta\Phi_{-I} = \widetilde{m}_{-I}\Phi_{-I}, \quad \delta\Phi^{\dagger}_{-I} = -\widetilde{m}_{-I}\Phi^{\dagger}_{-I}$$

Lattice Action: Q-exact form

$$S^{(\text{lat})}_{\text{mat}} = \sum_{I=1}^{n}\left[S^{(\text{lat})}_{\text{mat},+I} + S^{(\text{lat})}_{\text{mat},-I}\right]$$

$$\begin{aligned}
S^{(\text{lat})}_{\text{mat},+I} = Q\sum_x \Bigg[&\frac{1}{2}\bar{\psi}_{+IL}(x)\Bigg\{\left(\frac{1}{2}(D_1+D_1^*)+i\frac{1}{2}(D_2+D_2^*)\right)\phi_{+I}(x) - F_{+I}(x) \\
&\qquad + \sum_\mu \frac{r}{2}(D_\mu - D_\mu^*)\phi_{-I}(x)^\dagger\Bigg\} \\
&+\frac{1}{2}\Bigg\{\left(\frac{1}{2}(D_1+D_1^*)-i\frac{1}{2}(D_2+D_2^*)\right)\phi_{+I}(x)^\dagger - F_{+I}(x)^\dagger \\
&\qquad + \sum_\mu \frac{r}{2}(D_\mu - D_\mu^*)\phi_{-I}(x)\Bigg\}\psi_{+IR}(x) \\
&+\frac{1}{2}\bar{\psi}_{+IR}(x)(\bar{\phi}(x)-\widetilde{m}^*_{+I})\phi_{+I}(x) - \frac{1}{2}\phi_{+I}(x)^\dagger(\bar{\phi}(x)-\widetilde{m}^*_{+I})\psi_{+IL}(x) \\
&+ i\phi_{+I}(x)^\dagger\chi(x)\phi_{+I}(x)\Bigg], \qquad (3.6)
\end{aligned}$$

$$S^{(\text{lat})}_{\text{mat},-I} = Q\sum_x\left[\frac{1}{2}\left\{\left(\frac{1}{2}(D_1+D_1^*)+i\frac{1}{2}(D_2+D_2^*)\right)\phi_{-I}(x) - F_{-I}(x)\right.\right.$$

$$
\begin{aligned}
&\qquad\qquad +\sum_{\mu}\frac{r}{2}(D_\mu - D^*_\mu)\phi_{+I}(x)^\dagger\Big\}\,\bar\psi_{-IL}(x)\\
&+\frac{1}{2}\psi_{-IR}(x)\left\{\left(\frac{1}{2}(D_1+D^*_1) - i\frac{1}{2}(D_2+D^*_2)\right)\phi_{-I}(x)^\dagger - F_{-I}(x)^\dagger\right.\\
&\qquad\qquad \left. +\sum_{\mu}\frac{r}{2}(D_\mu - D^*_\mu)\phi_{+I}(x)\right\}\\
&+\frac{1}{2}\psi_{-IL}(x)(\bar\phi(x) - \widetilde{m}^*_{-I})\phi_{-I}(x)^\dagger - \frac{1}{2}\phi_{-I}(x)(\bar\phi(x) - \widetilde{m}^*_{-I})\bar\psi_{-IR}(x)\\
&\left.- i\phi_{-I}(x)\chi(x)\phi_{-I}(x)^\dagger\right], \qquad (3.7)
\end{aligned}
$$

Superpotential terms: (i: gauge group index)

$$
\begin{aligned}
S^{(\mathrm{lat})}_{\mathrm{pot}} = Q\sum_x\sum_I\sum_{i=1}^{N}\Bigg[&-\frac{\partial W}{\partial\phi_{+Ii}(x)}\psi_{+IRi}(x) - \frac{\partial W}{\partial\phi_{-Ii}(x)}\bar\psi_{-IRi}(x)\\
&-\bar\psi_{+ILi}(x)\frac{\partial\overline{W}}{\partial\phi^*_{+Ii}(x)} - \psi_{-ILi}(x)\frac{\partial\overline{W}}{\partial\phi^*_{-Ii}(x)}\Bigg] \qquad (3.8)
\end{aligned}
$$

Note

Due to the Wilson term, the flavor symmetry of $S_{\rm mat}^{(\rm lat)}$ is down to $U(1)^n$ (diagonal subgroup of $U(1)^n \times U(1)^n$).

$\Rightarrow$ The lattice action is Q-SUSY invariant only when $\widetilde{m}_{+I} = \widetilde{m}_{-I} (\equiv \widetilde{m}_I)$
(We can consider the setting of $\widetilde{m}^*_{+I} \neq \widetilde{m}^*_{-I}$!)

◇ $U(1)_A$-symmetry with the charges:

$$\begin{aligned} +2 &: \phi \\ +1 &: \psi_\mu, \quad \psi_{\pm IL}, \quad \bar{\psi}_{\pm IR} \\ -1 &: \chi, \quad \eta, \quad \psi_{\pm IR}, \quad \bar{\psi}_{\pm IL} \\ -2 &: \bar{\phi}, \\ 0 &: \text{the others} \end{aligned} \tag{3.9}$$

is realized in the lattice action when all the twisted masses are zero.
In particular, the Wilson terms are consistent with the $U(1)_A$-symmetry.

Note

$U(1)_A$ is not anomalous when $n_+ = n_-$.

$U(1)_A$-WT identity:

$$\partial_\mu \left\langle j_\mu^{U(1)_A}(x) \right\rangle = 2 \left\langle \sum_{I=1}^{n} \left(\mathcal{M}_{+I}(x) + \mathcal{M}_{-I}(x) \right) \right\rangle, \tag{3.10}$$

with

$$\begin{aligned} \mathcal{M}_{+I}(x) &= \widetilde{m}_I \left(\phi_{+I}(x)^\dagger \bar{\phi}(x) \phi_{+I}(x) + \bar{\psi}_{+IL}(x) \psi_{+IR}(x) \right) \\ &\quad - \widetilde{m}^*_{+I} \left(\phi_{+I}(x)^\dagger \phi(x) \phi_{+I}(x) + \bar{\psi}_{+IR}(x) \psi_{+IL}(x) \right) \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathcal{M}_{-I}(x) &= \widetilde{m}_I \left(\phi_{-I}(x) \bar{\phi}(x) \phi_{-I}(x)^\dagger + \psi_{-IR}(x) \bar{\psi}_{-IL}(x) \right) \\ &\quad - \widetilde{m}^*_{-I} \left(\phi_{-I}(x) \phi(x) \phi_{-I}(x)^\dagger + \psi_{-IL}(x) \bar{\psi}_{-IR}(x) \right). \end{aligned} \tag{3.12}$$

We can investigate the general case of $n_+ \neq n_-$, if the fields $\Phi_{+I}, \bar{\Phi}_{+I}$ $(I = n_+ + 1, \cdots, n)$ and $\Phi_{-I'}, \bar{\Phi}_{-I'}$ $(I' = n_- + 1, \cdots, n)$ are decoupled by sending $\widetilde{m}^*_{+I} \to \infty$ $(I = n_+ + 1, \cdots, n)$, $\widetilde{m}^*_{-I'} \to \infty$ $(I' = n_- + 1, \cdots, n)$.

Regarding $U(1)_A$-anomaly, we can check that the decoupling is achieved in the lattice perturbation,
and the anomalous WT-identity for n_+ fundamentals and n_-

anti-fundamentals is correctly obtained :

$$\partial_\mu \left\langle j_\mu^{U(1)_A}(x) \right\rangle = -\frac{1}{\pi}(n_+ - n_-) \mathrm{tr}\, F_{12}(x) + 2 \left\langle \sum_{I=1}^{n_+} \mathcal{M}_{+I}(x) + \sum_{I=1}^{n_-} \mathcal{M}_{-I}(x) \right\rangle . \tag{3.13}$$

The anomaly arises from

$$\sum_{I=1}^{n} \mathrm{tr}\, F_{12}(x)\, (ra)^2 \int_{-\pi/a}^{\pi/a} \frac{d^2 q}{(2\pi)^2} \hat{q}^2$$
$$\times \left(\hat{q}^2 \cos(aq_1)\cos(aq_2) - 2\bar{q}_1 \cos(aq_1) - 2\bar{q}_2^2 \cos(aq_2) \right) \left(\Delta_{+I}(q)^2 - \Delta_{-I}(q)^2 \right), \tag{3.14}$$

where the lattice spacing a is introduced,
and $\bar{q}_\mu = \frac{1}{a}\sin(aq_\mu)$, $\hat{q}_\mu = \frac{2}{a}\sin(aq_\mu/2)$,

$$\Delta_{\pm I}(q) = \frac{1}{\bar{q}^2 + \left(\frac{ra}{2}\hat{q}^2\right)^2 + \widetilde{m}_I \widetilde{m}^*_{\pm I}}. \tag{3.15}$$

Note

The decoupling is not so trivial, because the holomorphic parts $\widetilde{m}_I$ are kept finite.
It is interesting to clarify the cases that the decoupling holds.

4 Outlook

$\diamondsuit$ **We have presented a lattice formulation of 2D $\mathcal{N} = (2,2)$ SQCD with exactly keeping Q-SUSY.**

- **Gauge Group $G = U(N)$ (or $SU(N)$), Compact link variables $U_\mu(x)$**
- **In order to resolve the doubling of the matters, the lattice action is constructed in the case of the same number of the fundamental matters and anti-fundamental matters ($n_+ = n_-(= n)$).**
- **Regarding the $U(1)_A$-anomaly, the case $n_+ \neq n_-$ is achieved by decoupling in the lattice perturbation.**
  $\Rightarrow$ $n_+ \neq n_-$ case from the beginning?, Ginsparg-Wilson?
- **In the $G = U(N)$ case, the FI-term and the ϑ-term for the overall $U(1)$ can be introduced:**
  $S_{\mathrm{FI}}^{(\mathrm{lat})} = Q\kappa \sum_x \mathrm{tr}\, \chi(x)$
  (Gauge field configurations of nontrivial topology can be taken into account by the admissibility condition.)

◇ Applications

- **Computer simulation for SUSY breaking** (c.f. [Kanamori's Talk])
- **Lattice formulation of gauged linear sigma-models (⊃ SQCD models)**
  ⇒ Numerical analysis of phases of Calabi-Yau nonlinear sigma models
  [Witten, Hanany-Hori, Hori-Tong]