Instanton Effective Action in Deformed Super Yang-Mills Theories

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contents

1. Introduction

non-perturbative properties of supersymmetric gauge theory from string theory

- Effective field theory on D-brane $\rightarrow U(N)$ super Yang-Mills
- We turn on the constant closed string backgrounds.
  (NSNS B-field, RR fields)
  - Nekrasov formula for $\mathcal{N}=2$ super Yang-Mills
  - Dijkgraaf-Vafa theory for $\mathcal{N}=1$ supersymmetric gauge theory

  Closed string backgrounds play an important role in these cases.

★ Here we will consider $\mathcal{N}=2$ super Yang-Mills theories in the background of RR 3-form (with fixed $(2\pi\alpha')^{1/2}\mathcal{F}$).

[Billo-Frau-Fucito-Lerda, 2006]

– Consider D3-D(-1) system in type IIB on $\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$ with constant $\mathcal{F}^{\alpha\beta}$ from RR 3-form. $(\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2 \Rightarrow \mathcal{N}=2)$

– Calculate the disk amplitudes of D(-1) and mixed amplitudes of D3-D(-1) with/without insertion of $\mathcal{F}^{\alpha\beta}$.

– Taking $\alpha' \to 0$ limit with fixed $(2\pi\alpha')^{1/2}\mathcal{F}^{\alpha\beta}$ and integrating out the fields(ADHM moduli), Nekrasov formula is obtained.

calculation in D3 side $\Rightarrow$ deformed 4D action can be obtained.

$\begin{cases} \text{Can we rederive the above result from the deformed action?} \\ \text{Extension to the case of } \mathcal{N}=4 \text{ (and } \mathcal{N}=2^* \text{ etc.)} \end{cases}$

2. Deformation in $\mathcal{N}=2$ Super Yang-Mills

Procedure:

– Consider (fractional) D3-brane in type IIB on $\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$.
We forget about $\mathbf{Z}_2$-orbifolding for a moment ($\Rightarrow \mathcal{N}=4$).

– calculate the disk amplitude with/without the insertion of
self-dual "graviphoton" vertex operator (for constant $\mathcal{F}^{\alpha\beta AB}$)

$$V_{\mathcal{F}}(z,\bar{z}) = (2\pi\alpha')\,\mathcal{F}^{\alpha\beta AB}\left[S_\alpha(z)S_A(z)e^{-\frac{1}{2}\phi(z)}S_\beta(\bar{z})S_B(\bar{z})e^{-\frac{1}{2}\phi(\bar{z})}\right].$$

$S_\alpha(z)$, $S_A(z)$: spin field, ϕ: bosonized superconformal ghost,

α, β : 4D spinor indices, A, B : $SU(4)_R$ indices

- classification of the "graviphoton" field strength $\mathcal{F}^{\alpha\beta AB}$
  (S: symmetric, A: antisymmetric)

1. (S,S)-type (RR 5-form) $\mathcal{F}^{(\alpha\beta)(AB)} = \mathcal{F}^{\mu\nu abc}(\sigma_{\mu\nu})^{\alpha\beta}\left(\Sigma_{[a}\bar{\Sigma}_b\Sigma_{c]}\right)^{AB}$
2. (S,A)-type (RR 3-form) $\mathcal{F}^{(\alpha\beta)[AB]} = \mathcal{F}^{\mu\nu a}(\sigma_{\mu\nu})^{\alpha\beta}(\Sigma_a)^{AB}$
3. (A,S)-type (RR 3-form) $\mathcal{F}^{[\alpha\beta](AB)} = \mathcal{F}^{abc}\epsilon^{\alpha\beta}\left(\Sigma_{[a}\bar{\Sigma}_b\Sigma_{c]}\right)^{AB}$
4. (A,A)-type (RR 1-form) $\mathcal{F}^{[\alpha\beta][AB]} = \mathcal{F}^{a}\epsilon^{\alpha\beta}(\Sigma_a)^{AB}$

Remark $\mathcal{F}^{\mu\nu a}$ satisfies the self-dual condition.

$$\mathcal{F}^{\mu\nu a} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma}{}^{a},$$

- $\mathcal{N}=4$ deformation term $(C^{\mu\nu a} \propto (2\pi\alpha')^{1/2}\mathcal{F}^{\mu\nu a})$

$$\delta\mathcal{L} =$$

$$\mathrm{Tr}\left[g C^{\mu\nu a}\Big(i\varphi_a F_{\mu\nu} - \frac{1}{2}(\bar{\Sigma}_a)_{AB}\Lambda^A \sigma_{\mu\nu}\Lambda^B \Big) + \frac{g^2}{2} C^{\mu\nu a} C_{\mu\nu}{}^{b} \varphi_a \varphi_b \right].$$

- orbifolding

$$\mathcal{N}=4: \text{ type IIB on } \mathbf{R}^4 \times \mathbf{R}^6 \Rightarrow$$
$$\mathcal{N}=2: \text{ type IIB on } \mathbf{R}^4 \times \mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$$

We put N fractional D3-branes at the fixed point of the orbifold $\mathbf{R}^4/\mathbf{Z}_2$. In terms of the fields, orbifold projection is expressed by

$$\Lambda^A_\alpha = 0 \text{ for } A=3,4, \quad \varphi_a = 0 \text{ for } a = 3,4,5,6,$$

and only $C^{\mu\nu[12]}$ and $C^{\mu\nu[34]}$ are nonzero.

Under this reduction, $\mathcal{L}_{\mathcal{N}=4}$ becomes $\mathcal{L}_{\mathcal{N}=2}$. $(I=1,2)$

$$\mathcal{L}_{\mathcal{N}=2} = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\Lambda^{I\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}D_{\mu}\bar{\Lambda}_{I}^{\dot{\beta}} - D_{\mu}\varphi D^{\mu}\bar{\varphi}\right.$$
$$\left. -\frac{i}{\sqrt{2}}g\Lambda^{I}[\bar{\varphi},\Lambda_{I}] + \frac{i}{\sqrt{2}}g\bar{\Lambda}_{I}[\varphi,\bar{\Lambda}^{I}] - \frac{1}{2}g^{2}[\varphi,\bar{\varphi}]^{2}\right].$$

$\mathcal{N}=2$ deformation term ([Billo-Frau-Fucito-Lerda] for $\bar{C}=0$)

$\delta\mathcal{L} =$

$$\mathrm{Tr}\left[ig(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)F_{\mu\nu} + \frac{i}{\sqrt{2}}g\bar{C}^{\mu\nu}\Lambda^{I}\sigma_{\mu\nu}\Lambda_{I} + \frac{g^{2}}{2}(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)^{2}\right].$$

Here $C^{\mu\nu}$ and $\bar{C}^{\mu\nu}$ are defined by

$$C^{\mu\nu} = 2\sqrt{2}iC^{\mu\nu[12]}, \quad \bar{C}^{\mu\nu} = -2\sqrt{2}iC^{\mu\nu[34]}.$$

3. Instanton Solution of Deformed $\mathcal{N}=2$ Super Yang-Mills

- gauge field part

$$\begin{aligned}\mathcal{L}_{\mathrm{E}} &= \mathrm{Tr}\left[\tfrac{1}{2}(F^{-}_{\mu\nu})^2\right] + \cdots \\ &= \mathrm{Tr}\left[\tfrac{1}{2}\left(F^{+}_{\mu\nu} - ig(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)\right)^2\right] + \cdots ,\end{aligned}$$

where $F^{\pm}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma})$. This means that the (anti-)self-dual equations are

$$\begin{aligned} F^{-}_{\mu\nu} &= 0 \quad \text{for self-dual case,} \\ F^{+}_{\mu\nu} - ig(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi) &= 0 \quad \text{for anti-self-dual case.} \end{aligned}$$

• weak coupling expansion for self-dual case

$$A_\mu = g^{-1}A_\mu^{(0)} + g^1 A_\mu^{(1)} + \cdots\,, \quad F_{\mu\nu} = g^{-1}F_{\mu\nu}^{(0)} + g^1 F_{\mu\nu}^{(1)} + \cdots\,,$$
$$\Lambda^I = g^{-\frac{1}{2}}\Lambda^{(0)I} + g^{\frac{3}{2}}\Lambda^{(1)I} + \cdots\,, \quad \bar\Lambda_I = g^{\frac{1}{2}}\bar\Lambda_I^{(0)} + g^{\frac{5}{2}}\bar\Lambda_I^{(1)} + \cdots\,,$$
$$\varphi = g^0\varphi^{(0)} + g^2\varphi^{(1)} + \cdots\,, \qquad \bar\varphi = g^0\bar\varphi^{(0)} + g^2\bar\varphi^{(1)} + \cdots\,.$$

From this expansion, we obtain the self-dual instanton equation from the equation of motion.

• self-dual equation for leading-order fields ($\bar\Lambda_I^{(0)}$ is subleading.)

$$F_{\mu\nu}^{(0)-} = 0, \quad (\bar\sigma^\mu)^{\dot\alpha\beta} D_\mu \Lambda_\beta^{(0)I} = 0,$$
$$D^2\varphi^{(0)} + i\sqrt{2}\Lambda^{(0)I}\Lambda_I^{(0)} + iC^{\mu\nu}F_{\mu\nu}^{(0)} = 0,$$
$$D^2\bar\varphi^{(0)} + i\bar C^{\mu\nu}F_{\mu\nu}^{(0)} = 0.$$

- one-instanton solution in the case of SU(2) gauge group

$$(A_\mu^{(0)})_u{}^v = \frac{2i(\sigma_{\mu\nu})_u{}^v(x^\nu + a'^\nu)}{(x_\mu + a'_\mu)^2 + \rho^2}, \quad (\Lambda_\alpha^{(0)I})_u{}^v = \frac{8i\rho^2(\epsilon_{u\alpha}\xi^v + \delta_\alpha^v \xi_u)}{((x_\mu + a'_\mu)^2 + \rho^2)^2},$$

$$(\varphi^{(0)})_u{}^v = \frac{-8\sqrt{2}i\rho^2\xi_u^I\xi_I^v}{((x_\mu + a'_\mu)^2 + \rho^2)^2} + \frac{(x + a')_{ur}\phi^r{}_s(\bar{x} + \bar{a}')^{sv} + \rho^2 C_u{}^v}{(x_\mu + a'_\mu)^2 + \rho^2}.$$

$$(\bar{\varphi}^{(0)})_u{}^v = \frac{(x + a')_{ur}\bar{\phi}^r{}_s(\bar{x} + \bar{a}')^{sv} + \rho^2 \bar{C}_u{}^v}{(x_\mu + a'_\mu)^2 + \rho^2}.$$

moduli parameters

(bosonic) $-a'_\mu$: position, ρ : size.

(fermionic) $\xi_\alpha^I = \zeta_\alpha^I + x^\mu \sigma_{\mu\alpha\dot{\alpha}}\bar{\eta}^{I\dot{\alpha}}$,

ζ_α^I : moduli of SUSY transformation,

$\bar{\eta}^{I\dot{\alpha}}$: moduli of superconformal transformation.

• brief review on (super) ADHM construction

Introduce the $(N+2k)\times 2k$ matrix $\Delta_{\dot\alpha} = a_{\dot\alpha} + b^{\alpha}\sigma_{\mu\alpha\dot\alpha}x^{\mu} = (w_{\dot\alpha}\,,\ (a'+x)_{\alpha\dot\alpha})^{T}$ which satisfies the ADHM constraint

$$\bar\Delta^{\dot\alpha}\Delta_{\dot\beta} = f^{-1}\delta^{\dot\alpha}{}_{\dot\beta}\,, \quad f\,:\ x^{\mu}\text{-dependent } k\times k \text{ matrix,}$$

and $(N+2k)\times N$ matrix U which is the normalized zero-mode of $\bar\Delta$, i.e. $\bar\Delta U = 0$. Then the self-dual gauge field is constructed as

$$A^{(0)}_{\mu} = -i\bar U\partial_{\mu}U \quad \Rightarrow \quad F^{(0)}_{\mu\nu} = -4i\bar U b^{\alpha}(\sigma_{\mu\nu})_{\alpha}{}^{\beta}\bar b_{\beta}U\ :\ \text{self-dual.}$$

Dirac equation on the self-dual background is solved as $\Lambda^{(0)I}_{\alpha} = \bar U(\mathcal{M}^{I}f\bar b_{\alpha} - b_{\alpha}f\bar{\mathcal{M}}^{I})U$, where the constant $(N+2k)\times k$ matrix $\mathcal{M}^{I}$ satisfies the fermionic ADHM constraint

$$\bar\Delta^{\dot\alpha}\mathcal{M}^{I} + \bar{\mathcal{M}}^{I}\Delta^{\dot\alpha} = 0.$$

- solution of leading-order self-dual equation from ADHM

$$A_\mu^{(0)} = -i\bar{U}\partial_\mu U, \quad \Lambda_\alpha^{(0)I} = \bar{U}(\mathcal{M}^I f \bar{b}_\alpha - b_\alpha f \bar{\mathcal{M}}^I)U,$$
$$\varphi^{(0)} = i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{U}\mathcal{M}^I f \bar{\mathcal{M}}^J U + \bar{U}\begin{pmatrix} \phi & 0 \\ 0 & \chi \mathbf{1}_2 + \mathbf{1}_k C \end{pmatrix}U,$$
$$\bar{\varphi}^{(0)} = \bar{U}\begin{pmatrix} \bar{\phi} & 0 \\ 0 & \bar{\chi}\mathbf{1}_2 + \mathbf{1}_k \bar{C} \end{pmatrix}U,$$

where $\phi = \lim_{|x|\to\infty} \varphi^{(0)}$, $\bar{\phi} = \lim_{|x|\to\infty} \bar{\varphi}^{(0)}$, and

$$\chi = \boldsymbol{L}^{-1}\left(-i\tfrac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J + \bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}} + C^{\mu\nu}[a'_\mu, a'_\nu]\right),$$
$$\bar{\chi} = \boldsymbol{L}^{-1}\left(\bar{w}^{\dot{\alpha}}\bar{\phi}w_{\dot{\alpha}} + \bar{C}^{\mu\nu}[a'_\mu, a'_\nu]\right),$$
$$\boldsymbol{L} = \frac{1}{2}\{\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}}, *\} + \left[a'_\mu, [a'^\mu, *]\right].$$

Remark

1. When $\phi = \bar{\phi} = \bar{C}^{\mu\nu} = 0$, the solution

$$
\begin{aligned}
A_\mu &= A_\mu^{(0)}, & & \\
\Lambda_\alpha^I &= \Lambda_\alpha^{(0)I}, & \bar{\Lambda}_{I\dot{\alpha}} &= 0, \\
\varphi &= \varphi^{(0)}\Big|_{\phi=0}, & \bar{\varphi} &= \bar{\varphi}^{(0)}\Big|_{\bar{\phi}=0},
\end{aligned}
$$

is an exact solution of the equation of motion.

2. In the case of $\bar{C}^{\mu\nu} = 0$, the leading-order solution is reduced to the solution constructed by [Billo et. al.].

4. Instanton Effective Action

• definition

In the instanton background, the action can be evaluated as

$$S_{\mathrm{E}} = \frac{8\pi^2}{g^2}k + g^0 S_{\mathrm{eff}}^{(0)} + \mathcal{O}(g^2), \quad k: \text{ instanton number.}$$

This $S_{\mathrm{eff}}^{(0)}$ is called the instanton effective action. The explicit form is given by the substitution of the weak coupling expansion into the action and the result is

$$S_{\mathrm{eff}}^{(0)} = \int d^4x \, \mathrm{Tr}\Big[\big(D_\mu \bar{\varphi}^{(0)}\big) D^\mu \varphi^{(0)} + \frac{i}{\sqrt{2}}\Lambda^{(0)\alpha I}\big[\bar{\varphi}^{(0)}, \Lambda^{(0)}_{\alpha I}\big]$$
$$- i\big(C^{\mu\nu}\bar{\varphi}^{(0)} + \bar{C}^{\mu\nu}\varphi^{(0)}\big)F^{(0)}_{\mu\nu} + \frac{i}{\sqrt{2}}\bar{C}^{(\alpha\beta)}\Lambda^{(0)I}_{\alpha}\Lambda^{(0)}_{\beta I}\Big].$$

• instanton effective action in terms of ADHM moduli

$S_{\text{eff}}^{(0)}$ is expressed in terms of the ADHM moduli parameters by plugging the solution and integrating over the spacetime coordinates (This integration is very complicated). Then we have

$$S_{\text{eff}}^{(0)} = 4\pi^2 \text{tr}_k \Big[-i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mu}^I\bar{\phi}\mu^J + \frac{1}{2}\bar{w}^{\dot{\alpha}}(\bar{\phi}\phi + \phi\bar{\phi})w_{\dot{\alpha}} - \bar{\chi}\boldsymbol{L}\chi + i\frac{\sqrt{2}}{8}\bar{C}^{(\alpha\beta)}\epsilon_{IJ}\mathcal{M}'^I_\alpha\mathcal{M}'^J_\beta + \frac{1}{4}C^{\mu\nu}\bar{C}_{\mu\nu}\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}} \Big].$$

$S_{\text{eff}}^{(0)}$ can be also obtained in a different way, from the calculation of D3-D(-1) system by [Billo et. al.]. However, the last term of their result is $-\pi^2 C^{\mu\nu}\bar{C}_{\mu\nu}\text{tr}_k\left[a'_\rho a'^\rho\right]$.

This means we should have additional contribution to the original Lagrangian as

$$\mathcal{L}' = \mathrm{Tr}\Big[g^2 C_{\mu\rho}\bar{C}_{\nu\sigma}x^\rho x^\sigma F^{\mu\lambda}F^\nu{}_\lambda\Big]$$
$$\to -\pi^2 C^{\mu\nu}\bar{C}_{\mu\nu}\mathrm{tr}_k\big[a'_\rho a'^\rho + \bar{w}^{\dot{\alpha}}w_{\dot{\alpha}}\big]$$

Since this term depends on x^μ explicitly, we cannot calculate it from the disk amplitude of the open string.

Remark

1. $\mathcal{L}'$ does not contribute to the self-dual equation for leading-order fields. Then the instanton solution is unchanged by $\mathcal{L}'$.
2. $\mathcal{L} + \mathcal{L}'$ basically corresponds to the (truncated version of) super Yang-Mills Lagrangian on the Ω-background.

5. Summary and Outlook

Summary

1. We have constructed the (constrained) instanton solution of $\mathcal{N} = 2$ super Yang-Mills theory deformed by (S,A)-type RR 3-form.

2. We have also calculated the instanton effective action. But in order to compare with the result of the calculation of D3-D(-1) system, we should have additional contribution related with the Ω-background.

Outlook

1. $\mathcal{N} = 4$ (and $\mathcal{N} = 2^*$ etc.) case
   - The instanton solution can be obtained in a similar way.
   - The integration over x^μ can be performed and we obtain the similar result.

2. interpretation of the additional term
   - second-order interaction between D3-branes and RR 3-form (cf. argument from κ-symmetry)

3. instanton calculus (integration over ADHM moduli)
   - (deformed) BRST invariance, localization formula

4. deformation by other R-R fields ((A,S) 3-form, 5-form, 1-form)