Efficient and Precise Computation of Effective Action in Radial Backgrounds

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One-Loop Effective Action

Importance

- One-loop effective action: central object in QFT
  Partition function in STAT MECH
- Leading quantum effects in semiclassical approximations

Exact Results

- Euler-Heisenberg: Constant EM field
- Effective Potential: Constant scalar field
- Instanton Determinant for massless quark ('tHooft 76)
- Lower Dimensional Examples
- Difficult problem in General

Approximation Methods

- Small coupling expansion
- Derivative expansion ($\partial\phi$, ∂F, ∂R)
- Large mass expansion
- Numerical Computation
- Validity depends on the range of parameters

Our Method

- Spherical symmetry –> radial backgrounds (Separation of variables)
- Partial wave method (Numerical +Analytic)
- ‘Conventional’ regularization/renormalization
- Precision calculation
- Not many partial waves are needed

Sources

- J. Hur and H.Min, hepth-0805.079 (in press PRD)
- G.Dunne, C.Lee, J.Hur and H.Min, PRL94,072001; PRD71, 085019 (2005)
- G.Dunne, C.Lee, J.Hur and H.Min, PRD77,045004(2008)

- pair of partial differential operators

$$\mathcal{M} = -\partial^2 + V(r), \qquad \mathcal{M}^{\text{free}} = -\partial^2, \tag{1}$$

- log of functional determinant

$$\Gamma = \ln\left(\frac{\det[\mathcal{M}+m^2]}{\det[\mathcal{M}^{\text{free}}+m^2]}\right) = \sum_{l=0}^{\infty} g_l(d) \ln\left(\frac{\det[\mathcal{M}_l+m^2]}{\det[\mathcal{M}_l^{\text{free}}+m^2]}\right). \tag{2}$$

l: the angular momentum

- Degeneracy factor

$$g_l(d) = \frac{(2l+d-2)(l+d-3)!}{l!(d-2)!} \tag{3}$$

- Radial differential operator $\mathcal{M}_l$:

$$\mathcal{M}_l = -\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) + \frac{l(l+d-2)}{r^2} + V(r), \tag{4}$$

Theorem

the Gel'fand and Yaglom's theorem (1D)

$$\frac{\det(\mathcal{M}+m^2)}{\det(\mathcal{M}^{\text{free}}+m^2)} = \frac{\psi(\infty)}{\psi^{\text{free}}(\infty)}, \tag{5}$$

$\psi(r)$: *sol. of* $(\mathcal{M}+m^2)\psi = 0$, *i.c.* $\psi(0) = 0$; $\psi'(0) = 1$.

Example

Free 1D laplacian in a box

$$\frac{\det\left[-\frac{d^2}{dx^2}+m^2\right]}{\det\left[-\frac{d^2}{dx^2}\right]} = \prod_{n=1}^{\infty} \frac{\left[\left(\frac{n\,\pi}{L}\right)^2+m^2\right]}{\left[\left(\frac{n\,\pi}{L}\right)^2\right]} = \frac{\sinh(m\,L)}{m\,L} \tag{6}$$

$$\psi(x) = \frac{\sinh(m\,x)}{m}; \qquad \psi^{\text{free}}(x) = x$$

problems ($d \geq 2$)

- Determinants for all partial waves ($l = 0, 1, 2, \cdots$)
  Use the GY theorem for each partial wave
- the series is divergent:

$$\Gamma_l \sim \frac{1}{l}; \qquad g_l(d) \sim l^{d-2}$$

  Proper renormalization procedure
- slow rate of convergence
  Utilize the large-L expansion

Renormalized functional determinant

$$\Gamma_{\rm ren} = \Gamma + \delta\Gamma = \Gamma_{\rm L} + \Gamma_{\rm H}, \tag{7}$$

$$\Gamma_{\rm L} = \sum_{l=0}^{L} g_l(d) \ln\left(\frac{\det[\mathcal{M}_l + m^2]}{\det[\mathcal{M}_l^{\rm free} + m^2]}\right), \tag{8}$$

$$\Gamma_{\rm H} = \sum_{l=L+1}^{\infty} g_l(d) \ln\left(\frac{\det[\mathcal{M}_l + m^2]}{\det[\mathcal{M}_l^{\rm free} + m^2]}\right) + \delta\Gamma, \tag{9}$$

$\delta\Gamma$: the ‘conventional’ renormalization counterterm.

The low partial wave part $\Gamma_{\rm L}$:

- Calculate radial determinant for each l using the Gel’fand-Yaglom method.
- It grows $\sim L^{d-2}$

the high partial wave part $\Gamma_{\rm H}$:

- the sum of all (infinite number of) partial wave contributions with $l \geq L+1$,
- UV divergences: regularization / renormalization counter terms
- analytic calculation is possible

uniform asymptotic series in $\frac{1}{L}$

$$\Gamma_{\rm H} = \int_0^\infty dr \left(Q_{\log} + \sum_{n=2-d}^{\infty} Q_{-n} L^{-n} \right), \qquad (10)$$

- Q_{-n}'s may have an implicit L dependency of $O(L^0)$
- $Q_{\log}$ behaves as $O(\ln L)$ in the large L limit.
- The uniform nature makes the r integrals in (10) well-defined.

With $L \to \infty$ limit

$$\Gamma_{\rm ren} = \lim_{L\to\infty} \left[\Gamma_{\rm L} + \int_0^\infty dr \left(Q_{\log} + \sum_{n=0}^{d-2} Q_n L^n \right) \right]. \tag{11}$$

Example

Instanton determinant of a massless quark

$$\begin{aligned} \Gamma_{\rm ren} &= \lim_{L\to\infty} \left\{ \sum_{l=0,\frac{1}{2},\ldots}^{L} (2l+1)(2l+2) \ln\left(\frac{2l+1}{2l+2}\right) \right. \\ &\quad \left. +2L^2 + 4L - \frac{1}{6}\ln L + \frac{127}{72} - \frac{1}{3}\ln 2 \right\} \\ &= -\frac{17}{72} - \frac{1}{6}\ln 2 + \frac{1}{6} - 2\zeta'(-1) = \alpha(1/2) \qquad \text{t}'\text{Hooft} \end{aligned} \tag{12}$$

Rate of Convergence

- the rate of convergence is too slow $O(\frac{1}{L})$
  We have to calculate huge number of partial wave determinants.
- Utilize the $\frac{1}{L}$-suppressed terms in the large L WKB series.

With finite L

$$\Gamma_{\rm ren} = \Gamma_{\rm L} + \int_0^\infty dr \left(Q_{\log} + \sum_{n=0}^{d-2} Q_n L^n \right) + \int_0^\infty dr \sum_{n=1}^{N} Q_{-n} \frac{1}{L^n} + O\left(\frac{1}{L^{N+1}}\right) \quad (13)$$

truncated renormalized effective action

$$\Gamma_{\rm ren} = \Gamma^{(N)}(L) + O\left(\frac{1}{L^{N+1}}\right) \tag{14}$$

Efficiency

- N: the order of truncation.
- for a given value of the cutoff L:
  more accurate $\Gamma_{\rm ren}$ <— more $\frac{1}{L}$-suppressed terms.
- for a given accuracy:
  lower value of L <– including some $\frac{1}{L}$-suppressed terms.

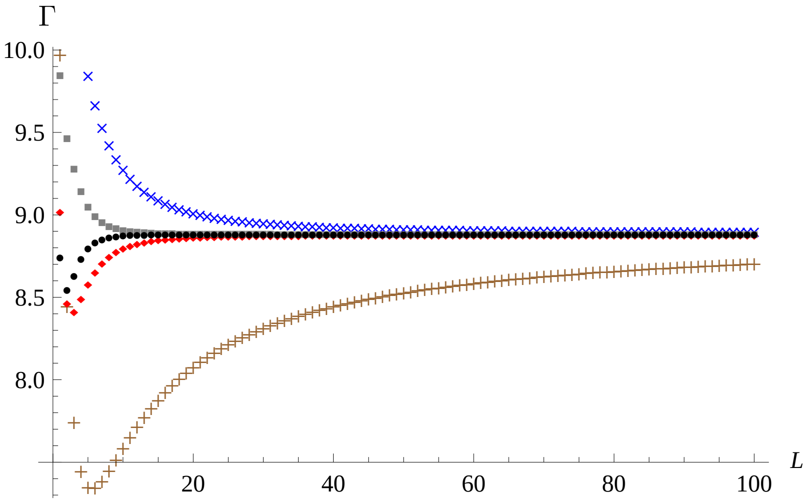


Figure: Plot of $\Gamma^{(N)}(L)$ with $N = 0$(brown), $N = 1$(blue) , . . . , $N = 4$(black).

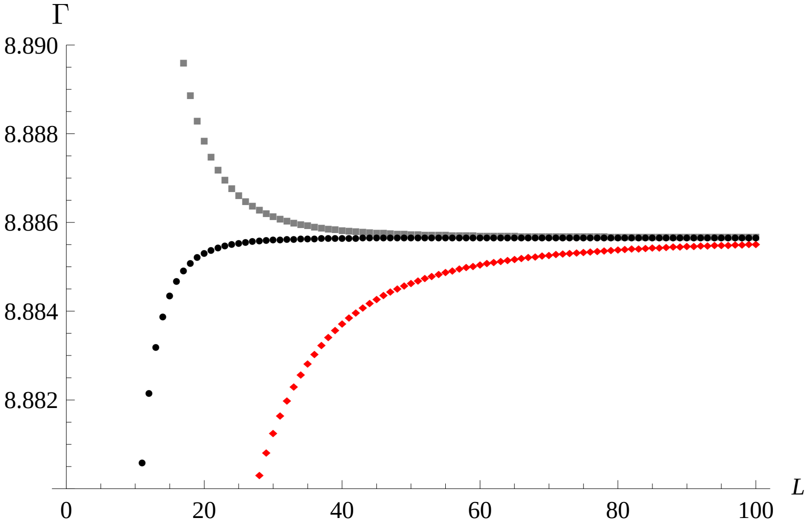


Figure: Same plot with FIG1 with magnification 250.

Proper-teim representation

- more convenient form of the determinant ratio

$$\frac{\det(\mathcal{M}_l + m^2)}{\det(\mathcal{M}_l^{\text{free}} + m^2)} = \frac{\det(\tilde{\mathcal{M}}_l + m^2)}{\det(\tilde{\mathcal{M}}_l^{\text{free}} + m^2)}, \quad (15)$$

$$\tilde{\mathcal{M}}_l \equiv r^{(d-1)/2} \mathcal{M}_l r^{-(d-1)/2} \quad (16)$$

$$= -\frac{d^2}{dr^2} + \frac{(l + \frac{d-3}{2})(l + \frac{d-1}{2})}{r^2} + V(r) \quad (17)$$

$$\equiv -\frac{d^2}{dr^2} + \mathcal{V}_l(r) \quad (18)$$

- The Schwinger proper-time representation

$$\ln \frac{\det(\tilde{\mathcal{M}}_l + m^2)}{\det(\tilde{\mathcal{M}}_l^{\text{free}} + m^2)} = -\int_0^\infty \frac{ds}{s}\, e^{-m^2 s} \times \int_0^\infty dr \left\{ \Delta_l(r, r; s) - \Delta_l^{\text{free}}(r, r; s) \right\}. \quad (19)$$

$$\Delta_l(r,r';s) = \langle r|e^{-s\tilde{\mathcal{M}}_l}|r'\rangle \tag{20}$$

satisfies the equation

$$\left\{\partial_s - \partial_r^2 + \mathcal{V}_l(r)\right\}\Delta_l(r,r';s) = 0. \tag{21}$$

large l and derivative expansion

rescale the potential $\mathcal{V}_l$, the proper-time s:

$$\mathcal{V}_l = l^2\mathcal{U}_l, \qquad s = \frac{t}{l^2}. \tag{22}$$

$$\left\{\partial_t - \frac{1}{l^2}\partial_r^2 + \mathcal{U}_l(r)\right\}\Delta_l\left(r,r';\frac{t}{l^2}\right) = 0. \tag{23}$$

the large l expansion $\sim$ the derivative expansion the proper time radial Green function at the same points:

$$\Delta(r,r;s) = \frac{e^{-s\mathcal{V}_l}}{\sqrt{4\pi s}}\Bigg[1+\lambda^2\left(\frac{s^3}{12}(\mathcal{V}_l')^2-\frac{s^2}{6}\mathcal{V}_l''\right)$$
$$+\lambda^4\left(\frac{s^6}{288}(\mathcal{V}_l')^4-\frac{11s^5}{360}(\mathcal{V}_l')^2\mathcal{V}_l''+\frac{s^4}{40}(\mathcal{V}_l'')^2+\frac{s^4}{30}\mathcal{V}_l'\mathcal{V}_l^{(3)}-\frac{s^3}{60}\mathcal{V}_l^{(4)}\right)$$
$$+\lambda^6\Bigg(\frac{s^9}{10368}(\mathcal{V}_l')^6-\frac{17s^8}{8640}(\mathcal{V}_l')^4\mathcal{V}_l''+\frac{83s^7}{10080}(\mathcal{V}_l'\mathcal{V}_l'')^2+\frac{s^7}{252}(\mathcal{V}_l')^3\mathcal{V}_l^{(3)}$$
$$-\frac{43s^6}{2520}\mathcal{V}_l'\mathcal{V}_l''\mathcal{V}_l^{(3)}-\frac{5s^6}{1008}(\mathcal{V}_l')^2\mathcal{V}_l^{(4)}+\frac{23s^5}{5040}(\mathcal{V}_l^{(3)})^2+\frac{19s^5}{2520}\mathcal{V}_l''\mathcal{V}_l^{(4)}$$
$$+\frac{s^5}{280}\mathcal{V}_l'\mathcal{V}_l^{(5)}-\frac{s^4}{840}\mathcal{V}_l^{(6)}\Bigg)+O\left(\lambda^8\right)\Bigg].$$

Γ_{H} has a finite expression

$$\Gamma_{\mathrm{H}} = \lim_{\epsilon\to 0}\Bigg[-\int_0^\infty \frac{ds}{s}\,(\mu^2 s)^\epsilon e^{-m^2 s}\int_0^\infty dr \sum_{l=L+1}^{\infty} g_l(d)\,\{\Delta_l(r,r;s) - \Delta_l^{\mathrm{free}}(r,r;s)\Big\} + \delta\Gamma\Bigg]. \quad (25)$$

l summation

$$\sum_{l=L+1}^{\infty} e^{-s(A_2 l^2 + A_1 l + A_0)}(c_0 + c_1 l + c_2 l^2 + \cdots). \quad (26)$$

the Euler-Maclaurin formula–> large L asymptotic

$$\sum_{l=L+1}^{\infty} f(l) = \int_L^\infty f(l)dl - \frac{1}{2}f(L) - \frac{1}{12}f'(L) + \frac{1}{720}f^{(3)}(L) + \cdots, \quad (27)$$

Large-L expansion of Γ_{H}

$$\sum_{l=L+1}^{\infty} g_l(d) \left\{ \Delta_l(r,r;s) - \Delta_l^{\text{free}}(r,r;s) \right\} = \sum_{n=2-d}^{\infty} P_{-n} L^{-n}. \quad (28)$$

$$\Gamma_{\mathrm{H}} = \lim_{\epsilon \to 0} \left[- \int_0^{\infty} \frac{ds}{s} (\mu^2 s)^{\epsilon} e^{-m^2 s} \int_0^{\infty} dr \sum_{n=2-d}^{\infty} P_{-n} L^{-n} + \delta\Gamma \right]$$

$$= \sum_{n=2-d}^{\infty} P_{-n} L^{-n} \quad (29)$$

Fomular for P's in 2D

the degeneracy factor: $g_l(2) = (2 - \delta_{l0}$, ($l \geq 1$).
The large L expansion in (28) starts from P_0.

$$P_0 = -\frac{r}{2}\mathrm{erfc}\left[\frac{\sqrt{t}}{r}\right]V, \tag{30}$$

$$P_{-1} = \frac{e^{-\frac{t}{r^2}}}{2\sqrt{\pi}}\sqrt{t}V, \tag{31}$$

$$P_{-2} = \frac{e^{-\frac{t}{r^2}}}{6\sqrt{\pi}}\left\{\frac{t^{3/2}}{2r^2}(V - 2rV') - \frac{t^{5/2}}{r^4}V\right\} + \frac{\mathrm{erfc}\left[\frac{\sqrt{t}}{r}\right]}{12}t(3rV^2 - V' - rV''). \tag{32}$$

- One may perform t integration with these explicit forms of P_{-n}'s.
- the first term, P_0 —> a pole term:

$$-\int_0^\infty \frac{dt}{t} e^{-\frac{m^2 t}{L^2}} \left(\frac{\mu^2 t}{L^2}\right)^\epsilon \mathrm{erfc}\left(\frac{\sqrt{t}}{r}\right) = -\frac{1}{\epsilon} + \gamma_E - 2\ln\left(\frac{\mu r}{(1+u)L}\right)$$

$$u = \sqrt{1 + \frac{m^2 r^2}{L^2}}. \qquad (34)$$

- This pole $\frac{1}{\epsilon}$ is canceled by the counterterm $\delta\Gamma$.
- No more divergences in the limit $\epsilon \to 0$.
- Following integral formulas are useful:

$$\int_0^\infty \frac{dt}{t} t^n e^{-t\frac{u^2}{r^2}} = \frac{r^{2n}}{u^{2n}} \Gamma(n), \qquad (35)$$

$$\int_0^\infty \frac{dt}{t} t^n e^{-\frac{m^2 t}{L^2}} \mathrm{erfc}\left(\frac{\sqrt{t}}{r}\right) = 2\left(\frac{r}{2}\right)^{2n} \frac{\Gamma(2n)}{\Gamma(n+1)} \times {}_2F_1(n, n+1/2; n+1; 1-u^2), \qquad (36)$$

- Here note that we do not expand the function

$$u = \sqrt{1 + \frac{m^2 r^2}{L^2}}$$

- the uniform nature of our large L expansion.
- the first coefficient of the large L expansion:

$$Q_{\log} = \ln\left(\frac{\mu r}{(u+1)L}\right) rV, \tag{37}$$

next sub-leading terms:

$$Q_{-1} = -\frac{1}{2u} rV, \tag{38}$$

$$Q_{-2} = \frac{1}{24u^5(u+1)}\left\{-6r^3u^4V^2 + 2r^3u^4V'' + 2r^2(u^2+u+1)u^2V' \right. \\ \left. -r(u^3+u^2-3u-3)V\right\}, \tag{39}$$

$$Q_{-3} = \frac{1}{48u^7}\left\{6r^3u^4V^2 - 2r^3u^4V'' - 6r^2u^2V' - 3r(u^4 - 6u^2 + 5)V\right\}, \tag{40}$$

$$Q_{-4} = \frac{1}{1920L^4u^{11}(u+1)^2}\left\{80r^5(2u+1)u^8V^3 - 40r^5(2u+1)u^8(V')^2 \right.$$
$$+8r^5(2u+1)u^8V^{(4)} + 16r^4(2u^3+4u^2+6u+3)u^6V^{(3)} + 60r^3(u+1)^2(u^2-5)u^4V^2$$
$$-r(u+1)^2(81u^6 - 1185u^4 + 2695u^2 - 1575)V - 80r^4(2u^3+4u^2+6u+3)u^6VV'$$
$$+4r^2(4u^7+18u^6+32u^5-139u^4-310u^3+20u^2+350u+175)u^2V'$$
$$\left. -4r^3(4u^5+13u^4+22u^3-44u^2-110u-55)u^4V'' - 80r^5(2u+1)u^8VV''\right\}. \tag{41}$$

- See Ref: hep-th 0805.0079 (PRD in press) for other formulas in 3D, 4D, 5D.

The one-loop effective action:

$$\Gamma \sim \ln \frac{\det(-D^2 + m^2)}{\det(-\partial^2 + m^2)}, \tag{42}$$

Instanton-like profile

$$A_\mu(\mathbf{x}) = 2\eta_{\mu\nu a} x_\nu f(r) \frac{\tau^a}{2} \quad , \qquad (r \equiv |\mathbf{x}| = \sqrt{x_\mu x_\mu}) \tag{43}$$

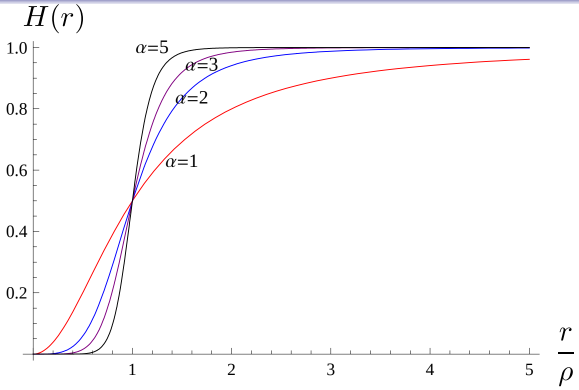
$$H(r) \equiv r^2 f(r) = \frac{1}{1 + (r/\rho)^{-2\alpha}}. \tag{44}$$

Radial operator for a partial wave of (j, l)

$$\tilde{\mathcal{M}}_{(l,j)} = -\partial_r^2 + \mathcal{V}_{(l,j)} \tag{45}$$

$$\mathcal{V}_{(l,j)} = \frac{(2l + \frac{1}{2})(2l + \frac{3}{2})}{r^2} + 4f(r)\left\{j(j+1) - l(l+1) - \frac{3}{4}\right\} + 3r^2 f(r)^2. \tag{46}$$

Gauge Theory Application



(a)

Figure: Plots of the radial profile function $H(r)$.

Partial wave expression

radial determinant

$$\Gamma_{(l,j)} = \ln\left(\frac{\det[\tilde{\mathcal{M}}_{(l,j)} + m^2]}{\det[\tilde{\mathcal{M}}_l^{\text{free}} + m^2]}\right), \qquad (47)$$

the low angular momentum part

$$\Gamma_{\mathrm{L}} = \sum_{l=0,\frac{1}{2},1,\cdots}^{L} (2l+1)(2l+2)\left\{\Gamma_{(l,l+\frac{1}{2})} + \Gamma_{(l+\frac{1}{2},l)}\right\} \qquad (48)$$

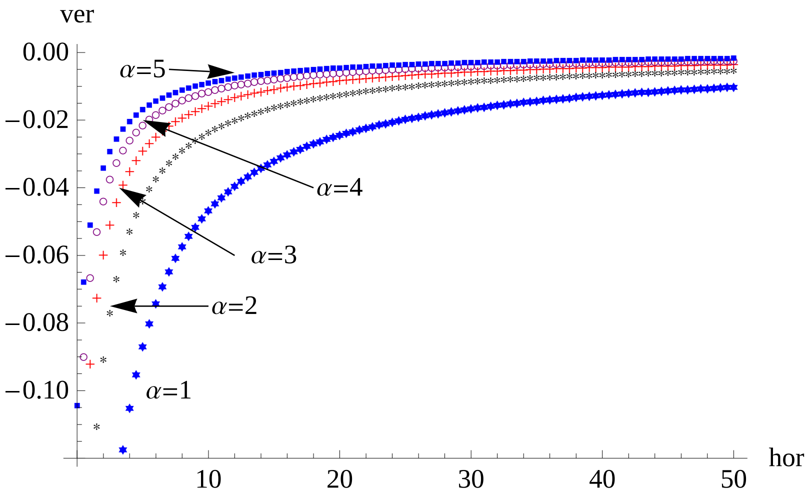


Figure: plot of partial wave determinant

With finite L

$$\Gamma_{\rm ren} = \Gamma_{\rm L} + \int_0^\infty dr \left(Q_{\log} + \sum_{n=0}^{d-2} Q_n L^n \right) + \int_0^\infty dr \sum_{n=1}^{N} Q_{-n} \frac{1}{L^n} + O\left(\frac{1}{L^{N+1}}\right) \quad (49)$$

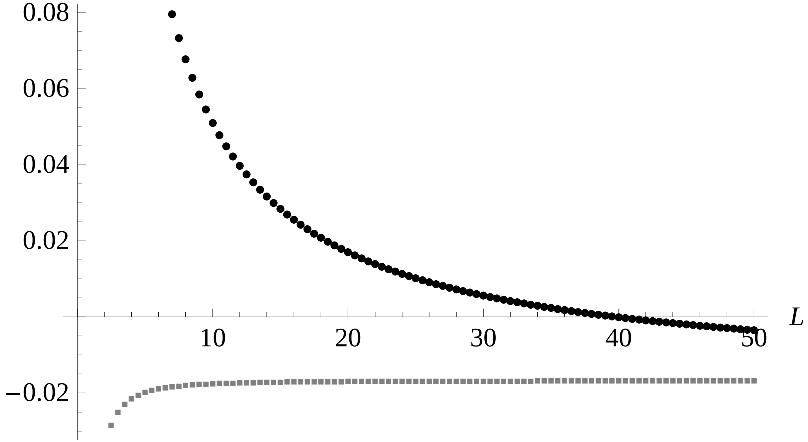
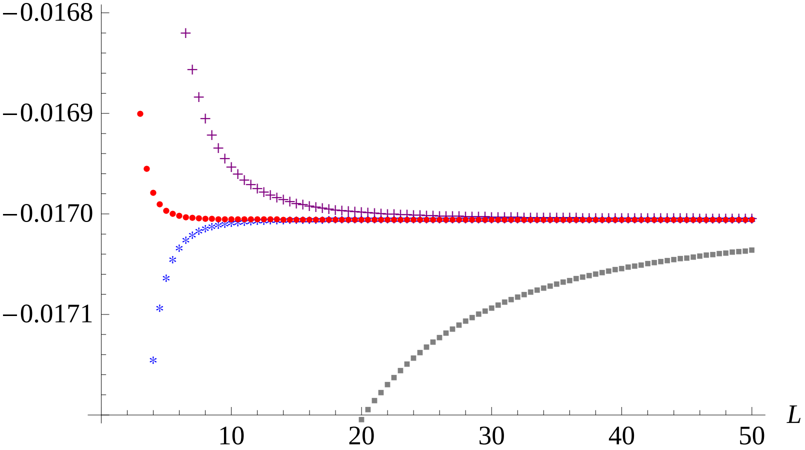


Figure: We have plotted L-dependence of the renormalized action

Gauge Theory Application



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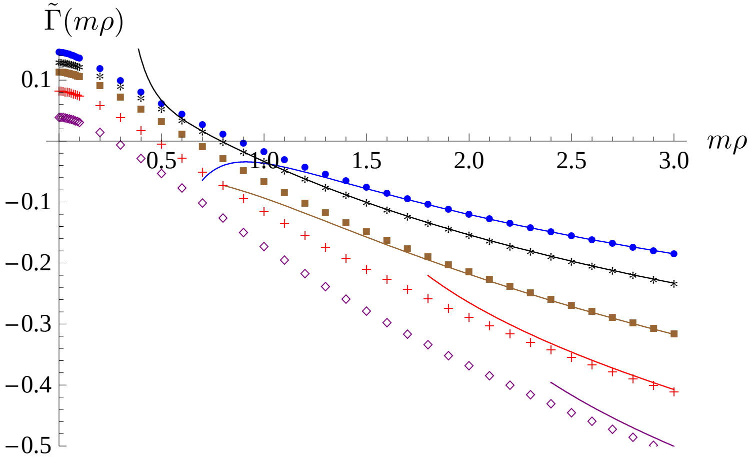

Figure: Plots of the modified effective action as a function of $m\rho$, for $\alpha = 1, 2, 3, 4, 5$

Small mass limit $\alpha = 1$

$$\tilde{\Gamma}(m) = A_1 + A_2\, m^2\, \ln m = 0.145873(29) + 0.499(17) m^2 \ln m \tag{50}$$

$$A_1 = -\frac{17}{72} + \frac{1}{6}(1 - \ln 2) - 2\zeta'(-1) \approx 0.14587331\cdots$$

$$A_2 = \frac{1}{2}$$

Table: Table of two parameters for $\alpha = 1, 2, 3, 4, 5$.

α	1	2	3	4	5
A_1	0.145873	0.129759	0.113632	0.082787	0.037283
A_2	0.499(17)	0.305(81)	0.291(68)	0.292(11)	0.295(03)

Conclusion

- We have developed an efficient computational method for the one-loop effective action in radial background fields.
- 1. Partial wave
  2. Partial wave cut off
  3. Gelfand-Yaglom method for some lowest partial waves
  4. analytic expression for large-L expansion for high angular momentum part
- Fermion determinant is under investigation